NOTES ON HARMONIC ANALYSIS PART II: THE FOURIER SERIES

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ABSTRACT. Fourier Series is the second of monographs we present on harmonic analysis. Harmonic analysis is one of the most fascinating areas of research in mathematics. Its centrality in the development of many areas of mathematics such as partial differential equations and integration theory and its many and diverse applications in sciences and engineering fields makes it an attractive field of study and research.

The purpose of these notes is to introduce the basic ideas and theorems of the subject to students of mathematics, physics, or engineering sciences. Our goal is to illustrate the topics with utmost clarity and accuracy, readily understandable by the students or interested readers. Rather than providing just the outlines or sketches of the proofs, we have actually provided the complete proofs of all theorems. This approach will illuminate the necessary steps taken and the machinery used to complete each proof.

The prerequisite for understanding the topics presented is the knowledge of Lebesgue measure and integral. This will provide ample mathematical background for an advanced under-

Clearly, $(\ ) = \cos\ + \sin$ **satisfies** $(\ ) = \chi(\cdot\ )$ **for all** $\in$.

Definition 1.2.

$$\mathbf{L^p(T)} = \{ f \quad n \quad on \ : \ |f|^p \ \sigma < \infty\},$$

$$|f|^p\ \sigma = \int_{-\pi}^{\pi} |f(\cdot\ )|^p \frac{}{2\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\ (\ )|^p\ .$$

Theorem 1.1.

$$\mathbf{L^p(T)} \supset \mathbf{L^r(T)}, \quad f\ p < \ , \qquad \|f\|_p \le \|f\|\ .$$

Proof: Using Hölder's inequality, we have: $(\ = \ /p > 1)$

$$|f|^p\ \sigma = |f|^p \cdot \mathbf{1}\ \sigma \le (\ (|f|^p)\ \sigma)^{1/}\ (\ 1\ '\ \sigma)^{1/\ '} = (\ |f|\ \sigma)^{p/} < \infty,$$

where $\frac{1}{} + \frac{1}{'} = 1$.

Definition 1.3. $f \in {}^1(\ )$ $n\ o$ o n f o o $n = \mathbf{0}, \pm\mathbf{1}, \pm\mathbf{2}, \cdots,$

$$_n(f) = f\chi^{-n}\ \sigma = \frac{1}{2\pi}\int_{\pi}^{\pi} (\ )^{-n}\ .$$

no f o n o

$$f(\cdot\ ) \sim \sum_{\infty}^{\sim\infty}$$

Proof: (1) is trivial. As to (2), let

$$f(\cdot) \sim \sum_{n=-\infty}^{\infty} {}_{n}(f)\, \cdot^{n}$$

be the Fourier series of f

Theorem 1.8. *If $f \in L^1(\mathbb{T})$ and $\frac{f(e^{ix})}{x}$ is integrable on $(-\pi, \pi)$, then $\sum_{n=-M}^{N} \hat{f}(n) \to 0$ as $M, N \to \infty$ independently.*

Proof: By the hypothesis,

$$g(e^{ix}) = \frac{f(e^{2ix})}{\sin x} \in L^1(\mathbb{T}).$$

(Note: the behavior of g near $\pm\pi$ is analogous to that of $\frac{f(e^{ix})}{x}$ near 0). Rewriting

$$f(e^{2ix}) = \frac{(e^{ix} - e^{-ix})\, g(e^{ix})}{2i}$$

and integrating against $\chi_{-2n}\, d\sigma$, we get

$$2i\, \hat{f}(n) = \hat{g}_{2n-1}(g) - \hat{g}_{2n+1}(g), \quad \forall n.$$

Hence, (telescoping sum), as $M, N \to \infty$,

$$2i \sum_{-M}^{N} \hat{f}(n) = \hat{g}_{-2M-1} - \hat{g}_{2N+1} \to 0.$$

(It is worth noting that the gist of the proof is considering $f(e^{2ix})$ and ending up with a telescoping sum.)

Corollary 1.2. *If $f \in L^1$ and f satisfies a Lipschitz condition of order α at x_0, then $\sum \hat{f}(n) e^{inx_0} \to f(e^{ix_0})$.*

Proof: Without loss of generality, we may assume that $x_0 = 0$ and $f(1) = 0$ and show that $\sum \hat{f}(n) \to 0$.

Assume that f satisfies the Lipschitz condition at x_0, that is, there is a neighborhood of x_0 so that for any x in that neighborhood, $|f(e^{ix}) - f(e^{ix_0})| \le K|x - x_0|^{\alpha}$ for some $0 < \alpha \le 1$. In our case of $x_0 = 0$ and $f(1) = 0$, this means that $|f(e^{ix})| \le K|x|^{\alpha}$, for x close to 0. Therefore, $\frac{f(e^{ix})}{x}$ is integrable on $(-\pi, \pi)$. Now the corollary follows from the above theorem.

Theorem 1.9. *(Principle of localization) If $f, g \in L^1(\mathbb{T})$ and $f =$*

Since $f -\ = 0$ on some interval, $f -$ satisfies Lipschitz condition at each interior point of that interval. Therefore,

$$\sum_{-} {}_{n}(f -)\chi^{n} \to 0.$$

This completes the proof.

Theorem 1.10. *ppo* $f(\cdot) \in {}^{1}(\ )$ *n* $\frac{f(\cdot)+f(-\cdot)}{}$, *n* *on* $(-\pi, \pi)$- *o*

$$\sum_{-} {}_{n}(f) \to 0 \qquad \to \infty.$$

Proof: Let $(\cdot)$ be such that

$$f(\ ^{2}\ ) - f(\ ^{-2}\ ) = \frac{1}{2}(\cdot\ -\ ^{-}\cdot\ )\ (\cdot).$$

Note that is integrable on $(-\pi, \pi)$ by the hypothesis. Integrating against $\chi^{-2n}\ \sigma$ we have

$$2(\ {}_{n}(f) - {}_{-n}(f)) = {}_{2n-1}(\ ) - {}_{2n+1}(\ ).$$

Adding up these equalities for $n = 0, \pm 1, \cdots, \pm$ we have

$$2\sum_{-} (\ {}_{n}(f) + {}_{-n}(f)) = 4\sum_{-} {}_{n}(f) = {}_{-2\ -1}(\ ) - {}_{2\ +1}(\ ) \to 0, \quad \text{as} \qquad \to \infty.$$

It is worth noting that, under the hypothesis of the theorem, it is not necessarily true that $\sum_{-} {}_{n}(f) \to 0$ as $\ ,\ \to \infty$ independently. For example, let $f(\cdot) = -1$ on $(-\pi, 0)$ and $= 1$ on $(0, \pi)$. Then ${}_{n}(f) = \frac{-}{n\pi}$ for n odd, $= 0$ for n even and

$$f(\cdot) \sim \sum \frac{1}{\pi}\left(\frac{1-(-1)^{n}}{n}\right).$$

Clearly, $\sum_{-} {}_{n}(f) = 0$ for all . However, if $= 2$ and $= 2$ then,

$$\sum_{-} {}_{n}(f)$$

Proof:

For 1 < p < ∞,

and

$$| \ | \le 2 \ {}_{|t|>\delta} K_n(t) \ \sigma(t) \to 0$$

as $n \to \infty$.

Now let $f \in L^p$ and $1 \le p < \infty$. For continuous f, the uniform convergence of $K_n * f$ to f implies convergence in L^p. Let $T_n(f) = K_n * f$. Then T_n a linear operator from $L^p(\mathbb{T}) \to L^p(\mathbb{T})$ such that $\|T_n(f)\|_p \le \|K_n\|_1 \|f\|_p$, i.e., $\|T_n\| \le 1$. Note that $T_n(f)$ converges in $L^p(\mathbb{T})$ for every $f \in C(\mathbb{T})$ and $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$. Therefore, Theorem 1.4 asserts that $T_n(f)$ converges in $L^p(\mathbb{T})$ for every $f \in L^p(\mathbb{T})$ and if we define $T(f) = L^p - \lim T_n(f)$ then T is a linear operator on L^p with bound ≤ 1. We prove that T is an identity on L^p. In fact, $T(f) = f$ for all $f \in C(\mathbb{T})$ and $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$. Let $f \in L^p$ and let $g_k \in C(\mathbb{T})$ with $g_k \to f$ in L^p. Then $T(f) = \lim T(g_k) = \lim g_k = f$ for all f, that is, $L^p - \lim K_n * f = f$ for all $f \in L^p$.

Theorem 2.7. (230Td.749463]TJR297.97011Tf7[(l)-3.05095(i)-3.0749463]TJbT

Let $_n$ be an approximate identity in ^{1}( ) so that $_n$'s are continuously differentiable. Then for every $f \in {}^1$, $_n * f$ is continuously differentiable. Let f be such that $_n(f) = 0$ for all n. Then for every n, $(_n * f) = {}_n( ) {}_n(f) = 0$ for all . By the first part of proof, $_n * f = 0$ everywhere. Thus, f, as a limit of $_n * f$ in ^{1}, is zero almost everywhere.

Corollary 3.1.

Theorem 3.5 (Unicity Theorem). μ *Bo* *on* *o* *o ffi* *n* *o* μ *n* *o*

$$\hat{\mu}(n) = {}_n(\mu) = \quad {}^{-n} \quad \mu(\ ).$$

$\hat{\mu}(n) = 0$ *o* n, n μ n

Proof: Note that if $\in C(\ )$ and ${}_n(\ ) = 0$ for all n then $= 0$. This is the weakest Unicity theorem of all. We use this version and the Riesz theorem to prove the strongest version as stated in the theorem.

For $\in C(\ )$, define the convolution

$$\mu * \ (\cdot \ ) = \quad (\ (\ - \ ) \ ) \ \mu(\ ).$$

We show that it is a continuous function. As usual, we write $(\cdot \ )$ as $(\ )$ for simplicity. Note that

$$
\begin{aligned}
|\mu * \ (\ + \ ) - \mu * \ (\ )| &= | \quad (\ (\ + \ - \ ) - \ (\ - \ )) \ \mu(\ )| \\
&\leq \ \| \ (\cdot + \ ) - \ (\cdot)\|_C \quad |\mu|(\ ) \\
&= \ \| \ (\cdot + \ ) - \ (\cdot)\|_C \|\mu\|.
\end{aligned}
$$

Since $|\mu|(\ )$ is finite and is uniformly continuous, the last expression tends to zero as $\to 0$.

The Fourier coefficients of $\mu *$ are ${}_n(\mu * \ ) = {}_n(\mu) \ (\ a$ C( ) its)) μ) eμ((

Proof: Define the functional $(\ ) = (\mu * (\nu * \ ))(1)$ on $C(\ )$. Clearly, it is linear. Observe that $\|\nu * \ \|_\infty \leq \|\nu\| \|\ \|_\infty$ for any measure $\nu \in \ (\ )$ and $\in C(\ )$. Applying twice, we have $|\ (\ )| \leq \|\mu\| \|\nu\| \|\ \|_\infty$. Therefore, is a continuous linear functional on $C(\ )$. By the Riesz theorem, there is $\gamma \in \ (\ )$ so that $(\ ) = \ (\ ) \ \gamma(\ ) = \gamma * \ (1)$ for all $\in C(\ )$. Then $\mu * \nu$ is defined to be the measure $\gamma \in \ (\ )$. Moreover, $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$.

Theorem 3.7. $(\ )$

which can be viewed as the value of $(\mu * \tilde{\mu}) * \;(\;,\;)$ at $\;= 0$.

4. THE C

which can be written as, if let $(n + \frac{1}{2})\ \ =\ \ ,$

$$2 \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin\ \ |}{\ } \quad .$$

We may disregard the parts of this integral over $(0, \pi)$ and $(n\pi, (n + \frac{1}{2}\pi)$, since the integrand is bounded. In view of the periodicity of sin , what remains can be written as

$$2 \int_\pi^{n\pi} \frac{|\sin\ \ |}{\ } \ = 2 \int_0^{\pi} \sin\ \ \Big(\sum_{=1}^{n-1} \frac{1}{\ + \ \pi} \Big) \quad .$$

For $0 \leq\ \ \leq \pi$, the sum is contained between $\frac{1}{\pi} \sum_{=2}^{n} \frac{1}{\ }$ and $\frac{1}{\pi} \sum_{=1}^{n-1} \frac{1}{\ }$, and so is strictly of order $\frac{1}{\pi} \ln n$. Collecting estimates, we obtain $\|\ \ _n\|_1 = \frac{4}{\pi^2} \ln n +\ \ (1)$.

Theorem 4.2. -- , on n o n on o o , , , po n-

Proof: Suppose it were true that $\ _n(\ ) = \sum_{-n}^{n}\ \ (\ )$ has a limit (i.e. $\ _n(\ )(\cdot)$ converges at $\ = 0$) for any $\ \in C(\ )$. Then we would have $|\ _n(\ )| \leq$ for all n, where is a constant. For each n, $\ _n$ is a linear functional on $C(\ )$, given as $\ _n(\ ) =$ 911.9552Tf4.439841.8Td[(()1.7603

When we sum the geometric series and simplify, we find

$$K_n(t) = \frac{1}{n}\Big(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\Big)^2.$$

Thus the Dirichlet and Fejér kernels are related by the formula

$$K_{2n+1}(t) = \frac{1}{2n+1} D_n^2(t).$$

Note that K_n is an approximate identity on $\mathbb{T}$. Thus for any $f \in L^1(\mathbb{T})$, $K_n * f(t) \to f(t)$ at every point of of continuity of f, and the convergence is uniform over every closed interval of continuity. In particular, $K_n * f$ tends to f uniformly everywhere if f is continuous everywhere. It holds also that if $f \in L^p$, $1 \le p < \infty$, then $\|K_n * f - f\|_p \to 0$.

The functions K_n are trigonometric polynomials; this fact has interesting consequence.

(1) Since K_n's are infinitely differentiable, any continuous function f is approximated uniformly by the infinitely differentiable functions (in fact, trigonometric polynomials) $K_n * f$.
(2) We also obtain another proof of the Unicity theorem in $L^1(\mathbb{T})$. Suppose that $\hat{f}(n) = 0$ for all n. Then for each n, $\widehat{(K_n * f)}(m) = \hat{K}_n(m)\hat{f}(m) = 0$, $\forall m$. Thus the trigonometric polynomial $K_n * f \equiv 0$. Since $\|K_n * f - f\|_1 \to 0$, $f = 0$ a.e.

The Poisson kernel

Define, for $0 < r < 1$,

$$P(r, t) = \sum_{-\infty}^{\infty} r^{|n|} e^{int}.$$

The series converges absolutely, and we can easily obtain that, if $z = re^{i\theta}$, $0 \le r < 1$, then

$$(e^{i(\theta - t)})$$

$f \in L^1(\,)$. [illegible] $* f(\,) \to f(\,)$ [illegible] $\uparrow 1$ [illegible] $* f(\,e^{i\theta}) \to f(\,)$ [illegible] $e^{i\theta} \to$ [illegible]

Proof: We prove that $*f(\,)$ is harmonic in (open unit disk). If f is real, then $*f$ is the real part of

$$\int \frac{\;+\;}{\;-\;} f(\,)\, \sigma(\,),$$

which is an analytic function of $= e^{i\theta}$ in . Hence $*f(\,e^{i\theta})$ is harmonic in . Since linear combinations of harmonic functions are harmonic, $*f(\,e^{i\theta})$ is a complex harmonic function on for any $f \in L^1(\,)$, the class of all complex, Lebesgue integrable functions on .

Theorem 4.4. [illegible] $f \in L^1(\,)$ [illegible] $f \geq 0$. [illegible]

Proof: Let $(\,) = *f(\,)$. Then $(\,)$ is harmonic in such that $\lim_{\uparrow 1} (\,e^{i\theta}) = f(\,e^{i\theta})$ for a.e. θ. Since is nonnegative, $(\,)$ is certainly positive whenever f is nonnegative. If $|f| \leq$, then $\| *f\|_\infty \leq \| \|_1 =$.

Theorem 4.5. [illegible]

Proof: We need only to show the necessity. Let be harmonic and bounded in . Let $_n$ th$\rangle \leq$

This can be verified by considering the representation theorem of harmonic functions in disk : If is real, continuous on $| \, | \leq \rho$ and harmonic in $| \, | < \rho$, then for $= \rho_1 e^{i\theta}$, $\rho_1 < \rho$,

$$(\rho_1 e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} [\frac{\rho e^{it} +}{\rho e^{it} -}] (\rho e^{it}) \, dt.$$

Let $\rho_1 = \rho$ (Note $0 \leq < 1$). Then

$$(\rho e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} [\frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}}] (\rho e^{it}) \, dt = \frac{1}{2\pi} \int_0^{2\pi} (e^{i(\theta - t)}) (\rho e^{it}) \, dt.$$

For complex harmonic, we consider it as a sum of real part and imaginary part.

5. Summability; Metric Theorems

We have shown the following theorems in the last section:

Theorem 5.1. [illegible] $(\ast f)(e^{i\theta})$ [illegible] $f \in L^1(T)$ [illegible]

[illegible] $f \in C(T)$, [illegible] $\ast f$ [illegible] $\to 1$.
[illegible] $f \in L^p(T)$, $1 \leq p < \infty$, [illegible] $\| \ast f - f \|_p \to 0$ [illegible] $\uparrow 1$.
[illegible] $f \in L^1(T)$.

Theorem 5.3. *A on n on n op n o n n on o on n d o o n n on on .*

Proof: We need only to show the necessity. Let be harmonic and bounded in . Let $_n \uparrow 1$ and write $_n(\;) = (\;_n\;)$. The sequence $_n$ is a bounded sequence in $^\infty($); hence for some sequence $n_J \to \infty$, $_{n_J}$ converges in the weak-* topology ($^\infty($

Proof: Suppose that is real. Since is a simply connected region, has a harmonic conjugate so that $=\ +$ is analytic in . We write $(\ ) = \sum_{n=0}^{\infty}\ _n\ ^n$. Then

$$(\ ) = (\ ) = {}_0 + \frac{1}{2}\left(\sum_{n=1}^{\infty} {}_n\ {}^n\ {}^{n\theta} + \sum_{n=1}^{\infty} \overline{}_n\ {}^n\ {}^{-n} \right)$$

$$= {}_0 + \frac{1}{2} \sum_{n=-\infty}^{\infty} {}_n\ {}^{|n|}\ {}^{n\theta}$$

where $_{-n} = \overline{}_n$ for $n = 1, 2, \cdots$. If is complex, then it is linear combination of two real for each $\in C($). In particular, for each n, $_n($ $_J) \to$ $_n(\mu)$ as $J \to \infty$. On the other hand, $_n($ $) =$ $_n$ $^{|n|} \to$ $_n$ as $\uparrow 1$. Therefore, $_n(\mu) =$ $_n$ for all n.

It follows from the Unicity theorem that μ is uniquely determined by $_n$, therefore by , and that since $_n($ $) =$ $_n$ $^{|n|} =$ $_n\mu$ $^{|n|} =$ $_n($ $* \mu)$, $=$ $* \mu$, i.e. $($ $\cdot$ $) =$ $* \mu(\cdot)$.

We show that $\|\mu\| = \lim_{\uparrow 1} A$. Note that $\mu = \lim_{J\to\infty}$ $($ $_J \cdot$ $)$ $\sigma($ $)$ in the weak* topology of $($ $)$ as the dual of $C($ $)$. It follows that $\|\mu\| \le \liminf_{J\to\infty} A$ $_J$ where A $_J = \|$

Let be a point where the above limit holds, i.e., at

$$\int_0 (f(\ + \ ) + f(\ - \ ) - 2f(\ )) \ = o(\ ).$$

Let $(\ ) = \int_0 (f(\ + \ ) + f(\ - \ ) -$

p_n … If $\lim_{t\to x\pm} f(t) = f(x\pm)$, then

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x-t)\,p_n(t)\,d\sigma(t) \to \frac{f(x+)+f(x-)}{2}.$$

Proof: If p_n is even, then

$$A_n = \int_{-\pi}^{\pi} f(x-t)\,p_n(t)\,d\sigma(t) - \frac{f(x+)+f(x-)}{2}$$

$$= \int_{0}^{\pi}\left(f(x+t)+f(x-t)-f(x+)-f(x-)\right)p_n(t)\,d\sigma(t).$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 < t \le \delta$, $|f(x+t)-f(x+)| < \varepsilon$ and $|f(x-t)-f(x-)| < \varepsilon$. We write

$$A_n = \left(\int_0^{\delta} + \int_{\delta}^{\pi}\right)\left(f(x+t)-f(x+)+f(x-t)-f(x-)\right)p_n(t)\,d\sigma(t) = A_{n,1}+A_{n,2}.$$

For $A_{n,1}$, we have

$$|A_{n,1}| \le 2\varepsilon\int_0^{\pi} p_n(t)\,d\sigma(t) = 2\varepsilon.$$

For $A_{n,2}$, we have

$$|A_{n,2}| \le 4M\int_{\delta}^{\pi} p_n(t)\,d\sigma(t) < \varepsilon,$$

for sufficiently large n.

An alternative proof of the above theorem.

Proof: First, we make the following assumptions successively:

(1) We may assume that $x = 0$ is the point where

$$\lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt = f(x).$$

That is,

$$\lim_{h\to 0}\frac{1}{2h}\int_{-h}^{h} f(t)\,dt = f(0).$$

Assume that the limit holds for f at $x = a$. Then $g(t) = f(t+a)$ satisfies

$$\lim_{h\to 0}\frac{1}{2h}\int_{-h}^{h} g(t)\,dt = g(0).$$

If the theorem is proved for g at 0, then $(p_n * g)(0) \to g(0)$ is simply $(p_n * f)(a) \to f(a)$.

(2) We may also assume that $f(0) = 0$. Let $g(t) = f(t) - f(0)$. Then $g(0) = 0$. If the theorem is proved for g, then $(p_n * g)(0) \to 0$ is simply $(p_n * (f(\cdot) - f(0)))(0) = (p_n * f)(0) - f(0) \to 0$, which is $(p_n * f)(0) \to f(0)$.

(3) Finally, we may assume that $\ell(\ )\ \sigma(\ ) = 0$. Let be a smooth function with $=\ \ell$, and vanishing on a neighborhood of $= 0$ (maintaining the above two

p p n

$$K_n^*( ) = \frac{2\pi^2 n}{1 + n^2 \; {}^2}.$$

Proof: The first formula for K_n gives $K_n( ) \le n$ (used for smaller). By Jordan's inequality, the second formula leads to $K_n( ) \le \frac{\pi^2}{n \; {}^2}$ (used for large). Combining these two gives $K_n( ) \le K_n^*( )$, where $K_n^*( ) = \frac{2\pi^2 n}{1+n^2 \; {}^2}$ (consider $| \; | \le \frac{1}{n}$ and $| \; | > \frac{1}{n}$ separately).

Theorem 6.2. $\in {}^1$ n po n

$$\lim_{\to 0} \frac{1}{} \int_0 | (+) + (-) - 2 | = 0$$

o o o $\in {}^1$

Proof: Let $\ = 0$ be a point where

$$\lim_{\to 0} \frac{\mu([- \ , \ ))}{2} = 0.$$

We show that

$$(p \ * \mu)(0) = \int_{-\pi}^{\pi} p \ (\ ) \ \mu(\ ) \to 0, \qquad \uparrow 1.$$

Let $(\ ) = \int_{-\pi} \mu(\ )$. Then, by Fubini's theorem,

$$\begin{aligned} \int_{-\pi}^{\pi} p'(\ ) \ (\ ) &= \int_{-\pi}^{\pi} p'(\ ) \int_{-\pi} \mu(\ ) \\ &= \int_{-\pi}^{\pi} \int^{\pi} p'(\ ) \ \mu(\ ) \\ &= \int_{-\pi}^{\pi} (p \ (\pi) - p \ (\ )) \ \mu(\ ) \\ &= p \ (\pi)\mu([-\pi, \pi)) + \int_{-\pi}^{\pi} p \ (\ ) \ \mu(\ ). \end{aligned}$$

It follows that

$$\int_{-\pi}^{\pi} p \ (\ ) \ \mu(\ ) = \int_{-\pi}^{\pi} p'(\ ) \ (\ ) \ - p \ (\pi)\mu([-\pi, \pi)).$$

For the last integral, noting that p' is odd, we have

$$\begin{aligned} \int_{-\pi}^{\pi} p'(\ ) \ (\ ) &= \int_{0}^{\pi} p'(\ )(\ (\ ) - \ (- \ )) \\ &= \int_{0}^{\pi} 2 \ p'(\ ) \frac{\mu([- \ , \ ))}{2} \\ &\to \lim_{\to 0} \frac{\mu([- \ , \ ))}{2} = 0, \qquad \uparrow 1. \end{aligned}$$

7. Herglotz' Theorem

Definition 7.1. *A o p n { $\}_{n=-\infty}^{\infty}$ po n*

$$\sum_{,n} \ _{-n} \ _n \geq 0$$

o y n $\{ \ _n\}_{n=-\infty}^{\infty}$ $\ _n = 0$ p o n n n.

Theorem 7.1. μ *y po on* $[0, 2\pi)$.

$$_n = \int \ ^{-,n} \ \mu(\ ).$$

so | (ϕ)| ≤ . Thus

Claim: A is a Banach algebra under multiplication (the product of f and g is defined as $f\bar{g}$) in the norm inherited from ℓ^1.

Proof: Let $f \in A$. $\|f\| = \|f\|_A = \|\{c_n(f)\}\|_{\ell^1}$. Then A is a normed linear space. We prove that A is complete. Let f_k be a Cauchy sequence in A, that is, $\{c_n(f_k)\}$ is a Cauchy sequence in ℓ^1. Assume that this sequence converges to $\{c_n\} \in \ell^1$. Let $f = \sum c_n e^{in\theta}$. Then $f \in A$ and $\|f_k - f\| = \|\{c_n(f_k) - c_n\}\|_{\ell^1} \to 0$.

To prove that A is a Banach algebra, we prove first that if both f and g are in A, then so is $f\bar{g}$. Note that if $f(\theta) = \sum c_k(f) e^{ik\theta}$ and $g(\theta) = \sum c_j(g) e^{ij\theta}$, then

$$f(\theta)\overline{g(\theta)} = \Big(\sum c_k(f) e^{ik\theta}\Big)\Big(\sum \overline{c_j(g)} e^{-ij\theta}\Big) = \sum_{k,j} c_k(f)\overline{c_j(g)} e^{i(k-j)\theta} = \sum d_l e^{il\theta},$$

where

$$d_l = \sum c_k(f)\overline{c_{k-l}(g)}.$$

If $c_k(f), c_j(g) \in \ell^1$, then d_l converges absolutely for every l. Moreover, since $\{d_l\}$ is the convolution of $\{c_k(f)\}$ and $\{\overline{c_j(g)}\} \in \ell^1$, $\{d_l\} \in \ell^1$ (like that in L^1) and $f\bar{g} \in A$. Secondly, we verify that $\|f\bar{g}\| \le \|f\|\|g\|$. Note that the inequality actually says that $\|a * b\|_{\ell^1} \le \|a\|_{\ell^1}\|b\|_{\ell^1}$ for $a, b \in \ell^1$. But it is true just like in L^1.

Claim: Define for all $\phi = \sum c_n e^{in\theta} \in A$,

$$L(\phi) = \sum c_n a_n.$$

Then $L(|\phi|^2) \ge 0$ for each $\phi \in A$.

We need to justify this definition first. Note that $|a_n| \le a_0$ for all n. Thus $\sum c_n a_n$ converges absolutely so that $L(\phi)$ is well-defined for all $\phi \in A$.

Next, we show $L(|\phi|^2) \ge 0$ for all $\phi \in A$. Note that $|\phi|^2 = \phi\bar{\phi} \in A$. By definition of L,

$$L(|\phi|^2) = \sum_{k,j} c_k(\phi)\overline{c_j(\phi)} a_{k-j}.$$

(The coefficient of a_0 is $\sum |c_k(\phi)|^2$). If $c_k(\phi)$'s are zeros except for finitely many k, then $\sum_{k,j} c_k(\phi)\overline{c_j(\phi)} a_{k-j} \ge 0$. Hence $L(|\phi|^2) \ge 0$ as long as the sum that evaluates $L(|\phi|^2)$ converges, which is indeed the case because a_n's are bounded.

Claim: If $\psi \in A$ and $\psi > 0$ (strictly positive!), then $\psi = |\phi|^2$ for some $\phi \in A$. Hence, $L(\psi) \ge 0$ for all $\psi \in A$ and $\psi > 0$.

Proof: Note that $\sqrt{\ }$ is analytic on the right-half (open) plane that contains the range of ψ ($\psi > 0$) and ψ

In addition, Bessel's inequality gives

$$\|\hat{f}\|_2 \le \|f\|_2.$$

Given p with $1 \le p \le 2$, let $0 \le \theta \le 1$ be such that

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}.$$

By the Riesz-Thorin theorem, we have

$$\|\hat{f}\|_{\ } \le \|f\|_p, \ \forall f \in \ _p,$$

where is given by

$$\frac{1}{\ } = 1 - \frac{1}{p}.$$

It is worth noting that we couldn't get the best constant in the Hausdorff- Young inequality. Beckner proved (Annals of Math, 102(1975)) for the Fourier transforms on that

$$\|\hat{f}\|_{\ } \le \sqrt{\frac{p^{1/p}}{\ ^{1/\ }}}\|f\|_p.$$

The following proof of the Hausdorff-Young inequality is due to A.P.Calderon and A. Zygmund. It suffices to show that for any trigonometric polynomial f with Fourier coefficients $\hat{\ } = (\hat{\ }_n)$ and $\|f\|_p = 1$ we have $\|\hat{\ }\|_{\ } \le 1$. Using the duality, we see that it suffices to show that

$$\Big| \sum \hat{\ }_n \ _n \Big| \le 1$$

for every sequence with $\| \ \|_p = 1$.

Put $f(\ ) = \ (\ )^{1/p} \ (\ )$ for $\ \in$ such that $(\ ) = |f(\ )|^p \ge 0$ and $|\ (\ )| = 1$. ($(\ ) = p\{\ \ (f(\ ))\}$. In case $f(\ ) = 0$, simply define $(\ ) = 1$). Similarly, put $\ _n = \ _n^{1/p} \ _n$ with $\ _n \ge 0$ and $|\ _n| = 1$.

Using

011[(U)-442.594(I)1.76236(n)-303.68(i)-3.05095(t)-3.04993(i)-3.04993(o)-1.874[(i)-3.04922(,)-221.749(w)-0.69822(,)-221

Pupre dn sing tnaimp(e)2.999438 1lamimpse

Since the sum has only finitely many terms, each one (as function of) is bounded in the strip $\frac{1}{2} \le$ ≤ 1. Hence ( ) is bounded in this strip with bound depending on $'_n$ and .

For $= 1$, we have

$$| \ (1 + ,)| \le \quad {}_n \quad ( ) \ = 1.$$

For $= \frac{1}{2}$, the Schwarz inequality gives

$$| \ (\frac{1}{2} + , \theta)| \le (\quad {}_n)^{1/2} (\quad | \quad ( )^{\frac{1}{2}+,\theta} \ ( ) \ ^{-,n} \ |^2)^{1/2}.$$

For any $\in {}^{p}$, by the Hölder and Hausdorff-Young inequalities,

$$|\int_{-\pi}^{\pi} (\ )\overline{\ _n(\ )}\ \sigma| = |\sum_{=-n}^{n} \hat{\ }(\ )\bar{\ }|$$

$$\leq (\sum_{=-n}^{n} |\hat{\ }(\ )|\ )^{1/}\ (\sum_{=-n}^{n} |\ |^{p})^{1/p}$$

$$\leq \|\bar{\ }\|\ \|\ \|_p \leq \|\ \|_p \|\ \|_p.$$

This implies that $\|\ _n\| \leq \|\ \|_p$ for all n. Note that this is valid for any $\in {}^{p}$. We have

$$\|\ -\ _n\| = \|\sum_{n<|\ |\leq}\ \ \| \leq \sum_{n<|\ |\leq} |\ |^{p}.$$

Therefore, $_n$ is a Cauchy sequence in and hence there exists an $\in$ so that $\|\ _n - \ \| \to 0$. We simply define $(\ ^{*}\ )(\ ) = \ (\ )$. Note that * is an adjoint operator to , the finite Fourier transform, in the sense that

$$<\ (\ ),\ >=<\ ;\ ^{*}(\ )>,$$

for all $\in {}^{p}$ and $\in {}^{p}$, where $<\ (\ ),\ >= \sum \hat{\ }(\ )\bar{\ }$ and $<\ ;\ ^{*}(\ )>= \int (\ )\ (\ )$ with defined as the limit of $_n$.

Moreover, for each and for any $n > |\ |$, by Hölder's inequality,

$$|\hat{\ }(\ ) - \ | = |\int (\ (\ ) - \ _n(\ ))\ \ \sigma| \leq \|\ - \ _n\|\ .$$

Therefore, $\hat{\ }(\ ) = \ $.

Remarks:

(1) The case $p = 2$ is the theorem of Riesz-Fischer.
(2) The case $p = 1$, to every $\in {}^{1}$ we may assign the continuous function $(\ ) = \sum\ \ $. Since the series converges uniformly, $= \hat{\ }(\ )$ and $\|\ \|_C \leq \|\ \|_1$.
(3) The restriction of the theorem to $1 \leq p \leq 2$ is essential. For there is a sequence $\in$ for all > 2 and yet is not the finite Fourier transform of any function in ^{1}.

The series

$$\sum \pm n^{-1/2} \cos n\ ,$$

with a suitable choice of signs, is a desired example as shown by the following theorem: If $\sum(\ _n^2 + \ _n^2)$ diverges, then almost all the series

$$\sum\ _n(\ )(\ _n \cos n\ + \ _n \sin n\ )$$

are not Fourier series (because almost all the series are almost everywhere non-Fejér summable).

Theorem 8.4. *on ... o ff Yo n n ... o* $1 \le p \le 2$, *n ... o ... on n o ... n on* $\in C$ *n* $\in$ *o ...* $p > 0$ ... $\|\hat{f}\| = \infty$ *o ...* < 2. *o ... po ... o o* $p > 2$ *o ...* $\|\hat{f}\| \le \|f\|_p$ *o* $\in$ $_p$.

Proof: **The construction of the desired function follows from the following theorem (see**

We now turn to the general case. Put

$$g(z) = M_0^{1-z} M_1^{z},$$

where $M^{\zeta} = \exp\{\zeta \log M\}$ for complex ζ. Then $g(z)$ is entire, g has no zero, $1/g$ is bounded in the closed strip,

$$|g(iy)| = M_0, \qquad |g(1+iy)| = M_1,$$

and hence f/g satisfies our previous assumptions. Thus $|f/g| \le 1$ in the strip, and this gives $|f(\theta + iy)| \le M_0^{1-\theta} M_1^{\theta}$ for all $0 \le \theta \le 1$.

An Alternative Proof:

Let $\varepsilon > 0$ and $\lambda \in \mathbb{R}$. Define

$$f_\varepsilon(z) = \exp\{\varepsilon z^2 + \lambda z\} f(z).$$

Then

$$f_\varepsilon(z) \to 0, \text{ as } y \to \pm\infty$$

and

$$|f_\varepsilon(iy)| \le M_0, \ |f_\varepsilon(1+iy)| \le M_1 e^{\varepsilon+\lambda}.$$

By the Phragmen-Lindelöf principle we therefore obtain

$$|f_\varepsilon(z)| \le \max\{M_0, M_1 e^{\varepsilon+\lambda}\}.$$

That is,

$$|f(\theta + iy)| \le \exp\{-\varepsilon(\theta^2 - y^2)\} \max\{M_0 e^{-\theta\lambda}, M_1 e^{(1-\theta)\lambda+\varepsilon}\}.$$

This holds for any fixed θ and y. Letting $\varepsilon \to 0$ we conclude that, if $\rho = \exp\{\lambda\}$,

$$|f(\theta + iy)| \le \max\{M_0 \rho^{-\theta}, M_1 \rho^{1-\theta}\}.$$

9. A Theorem of Minkowski

Let

$$\mathbb{T}^2 = \{(e^{2\pi i x}, e^{2\pi i y}) : x, y \in \mathbb{R}\}.$$

$\mathbb{T}^2$ is called the 2-dimensional torus, which is the Cartesian product of the unit circle $\mathbb{T} = \{e^{2\pi i x} : x \in \mathbb{R}\}$.

Let (m, n) be a lattice (integer coordinates) point in the plane and let $f(x, y)$ be a summable function on the unit square

$$Q = \{(x, y) : 0 < x < 1, 0 < y < 1$$

We may prove the Parseval relation on

The last equality is simply the result of change of variables. For the one above the last equality, we denote by $S_{-k,-n}$ the square with the lower left corner $(-k,-n)$. Then we have:

$$
\begin{aligned}
&\int\int_{\mathbb{R}^2} φ(2x,2y)\, e^{-2πi(xξ+yη)}\,dx\,dy \\
&= \sum_{k,n}\int\int_{S_{-k,-n}} φ(2x,2y)\, e^{-2πi(xξ+yη)}\,dx\,dy \\
&= \sum_{k,n}\int\int_{S} φ(2(x-k),2(y-n))\, e^{-2πi(xξ+yη)}\,dx\,dy \\
&= \int\int_{S}\sum_{k,n} φ(2(x-k),2(y-n))\, e^{-2πi(xξ+yη)}\,dx\,dy .
\end{aligned}
$$

On the other hand, we calculate $\int\int_S |f(x,y)|^2\,dx\,dy$.

$$
\begin{aligned}
\int\int_S |f(x,y)|^2\,dx\,dy &= \int\int_S f(x,y)\sum_{k,n} φ(2(x-k),2(y-n))\,dx\,dy \\
&= \int\int_{\mathbb{R}^2} f(x,y)φ(2x,2y)\,dx\,dy \\
&= \int\int_{\mathbb{R}^2}\sum_{k,n} φ(2(x-k),2(y-n))φ(2x,2y)\,dx\,dy \\
&= \sum_{k,n}\int\int_{\mathbb{R}^2} φ(2(x-k),2(y-n))φ(2x,2y)\,dx\,dy \\
&= 2^{-2}\sum_{k,n}\int\int_{\mathbb{R}^2} φ(x-2k,y-2n)φ(x,y)\,dx\,dy \\
&= 2^{-2}\sum_{k,n}\int\int_{C} φ(x-2k,y-2n)\,dx\,dy .
\end{aligned}
$$

The Parseval relation gives rise to

$$
2^{-2}\left|\int\int_C φ(x,y)\, e^{-πi(xξ+yη)}\,dx\,dy\right|^2 == 2^{-2}\sum_{k,n}\int\int_C φ(x-2k,y-2n)\,dx\,dy .
$$

If C contains no lattice point except the origin, then one can show that for $(x,y)\in C$ and $(k,n)\neq(0,0)$, $(x-2k,y-2n)\notin C$

Theorem 9.2. *C* *on* *o y* *n* *d* *o* $= 2$ *n y* *o* *o* *n* *n* *po n* $(, n) \neq (0, 0)$ *n C o on* *o n y*

Proof: Assume that, by a contradiction, $\overline{C}$ contains no lattice point other than the origin. We assume that C is bounded. Thus $\overline{C}$ is compact and there is $\delta > 0$ such that $(p, C) \geq \delta > 0$ for all lattice points p other than the origin. We may expand $\overline{C}$ slightly to a subset of so that is convex and symmetric about origin and yet contains no lattice point other than the origin. Since the volume of is > 2 , this is in contradiction to Minkowski's theorem.

It remains to show the construction of . Let

$$= \{ \in : (, C) \leq \frac{\delta}{2}\}.$$

We claim that if C is (closed) convex, then so is . Let and $y \in$ (Assume that they are not in C. Otherwise, nothing needs to be done.) Let $_0$ and $y_0 \in C$ such that $| - _0| = (, C)$ and $|y - y_0| = (y, C)$. For $0 \leq \lambda \leq 1$, we have $|(\lambda + (1 - \lambda) y) - (|\lambda _0 + (1 -$

If $| \quad - \quad | \leq$ **1 then** $\quad \geq$

where $^{n}(\ )$ is the nth power of with respect to ordinary multiplication. We prove this with $n = 2$. Using the Cauchy product of two series, we have

$$^{2}(\ ) \quad \sigma(\ ) = \quad ($$

Proof: (1). Prove that

where $\Phi(\ , \ \ , \ ) = \varphi(\ - \ \ ) + \varphi(\ ) - \varphi(\ - \ \ ) - \varphi($

Let $\psi \in \ell^1(Z^3)$ (sequence depending on three indices) be the Fourier transform of $\Psi(\; , \; , \;)$. Then ψ^{*n} is the Fourier transform of Ψ^n (with respect to ordinary multiplication) and $\|\psi^{*n}\|_{\ell^1(Z^3)} \leq$ [illegible] ^{4} for all n.

To see this, we first prove that

$$\|\mathcal{F}(\; {}^n\Phi(\; , \; , \;))\|_{\ell^1(Z^3)} \leq \text{[illegible]}^4, \qquad \forall n.$$

Note that

$$ {}^n\Phi(\; , \; , \;) = {}^n\varphi(\; - \;) \; {}^n\varphi(\;)$$

**Obviously, is a homomorphism of $A($ $)$ into $A($ $)$. Define a homomorphism of into ,
denoted by $'$, in such a way that**

$$'(\mathcal{F}) = \mathcal{F}(\ ())$$

for all $\in A($ $)$. This can be written as

$$'(\rho) = \square$$

Theorem 10.4.

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